A New Polytope for Symmetric Traveling Salesman Problem

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Abstract

The classical polytope associated with the symmetric traveling salesman problem (STSP) is usually studied as embedded in the standard subtour elimination polytope. Several classes of facet defining inequalities of the STSP polytope are used in practical enumeration algorithms. In this paper we consider the basic objects called *pedigrees* which are in 1-1 correspondence with the tours. The convex hull of these pedigrees yields a new polytope, called the *pedigree* polytope.

In this paper, we study the pedigree polytope as embedded in the MI-relaxation polytope introduced in an earlier work by the author. One of the interesting properties of these pedigree polytopes is that they can be defined in a recursive manner. This yields a sequence of flow problems and the feasibility of them is sufficient for the membership in the pedigree polytope under study. Other properties of the pedigree polytopes derived from these results and their consequences are studied.

1 Introduction

The traveling salesman problem (TSP) is about finding a minimum cost tour that starts from the home city and visits every city once and returns to the home city. A recent survey of TSP , with emphasis on computational achievements based on polyhedral combinatorics and facet defining inequalities appears in Jünger et.al. [12].

In 1954 Dantzig et.al. [10] formulated the asymmetric traveling salesman problem as a 0-1 linear program on a graph (V, E). Their formulation for the symmetric case gives rise to the standard subtour elimination polytope SEP_n.

Arthanari [2] posed the symmetric traveling salesman problem (STSP) as a multistage decision problem and gave a $0-1$ programming formulation of the same, involving three subscripted variables. The slack variables that arise out of this

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formulation are precisely the edge-tour incidence vectors. The multistage decision (dynamic programming) approach to TSP is not new. However the earlier formulations are different from that of [2]. The motivation for the multistage insertion formulation appears in [2], [6]. In [5] this formulation is shown to be equivalent to SEP_n, thereby reducing the number of constraints to a polynomial in n. Recently, Carr [9] has given an equivalent relaxation to SEP_n called *Cycle Shrink*. He also uses this relaxation to separate in polynomial time over certain classes of $STSP$ inequalities. It is known that the cycle shrink relaxation is conceptually same as the multi-stage insertion (MI) relaxation [7].

Arthanari [3] addresses the problem of checking whether a given $u \in SEP_n$ is in the $STSP$ polytope Q_n. There *polyhedral Star-inequalities* are defined and it is shown that if there exists a X such that (X, u) is feasible for the MI-relaxation and X satisfies the polyhedral star-inequalities then $u \in Q_n$.

In this paper we study an alternative polytope, called *pedigree* polytope associated with STSP. Section 2 introduces the preliminaries, notation and a flow feasibility problem called FAT problem. The pedigree polytope is next defined in section 3 and its connection to MI - formulation is studied in section 3.2. Characterisation of the pedigree polytope is attempted in section 4. The consequences of the results obtained in section 4, are discussed in the concluding section.

2 Preliminaries & Notations

This section gives a short review of the definitions and concepts from graph theory and flows in networks used in the paper.

Let R denote the set of reals. Similarly Q, Z, N denote the rationals, integers and natural numbers respectively, and B stands for the binary set of $\{0, 1\}$. Let R_+ denote the set of non negative reals. Similarly the subscript $_+$ is understood with rationals. Let R^d denote the set of d -tuples of reals. Similarly the superscript ^{d} is understood with rationals, etc. Let $R^{m\times n}$ denote set of $m \times n$ real matrices.

2.1 Graph Theory & Flows in Networks

Let n be an integer, $n \geq 3$. Let $V_n = \{v_1, \ldots, v_n\}$ be a set of *vertices*. Assuming, without loss of generality, that the vertices are numbered in some fixed order, we write, $V_n = \{1, \ldots, n\}$. Let $E_n = \{(i, j)|i, j \in V_n, i < j\}$ be the set of *edges / pairs*. The cardinality of E_n is denoted by $p_n = n(n-1)/2$. Let $K_n = (V_n, E_n)$ denote the complete graph with n vertices.

We denote the elements of E_n by e where $e = (i, j)$. We also use the notation ij for (i, j).

For a subset $F \subset E_n$ we write the *characteristic* vector of F by $x_F \in R^{p_n}$ where

$$x_F(e) = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

At times, we denote the characteristic vector by $\dot{F}$. If $F \subset E_r \subset E_n$, $r \leq n$. Then we also use $\dot{F}$ to denote the extension of $\dot{F}$ to R^{p_n} , by augmenting $\dot{F}$ with zero for

every $e \in E_n \setminus E_r$. We assume that the vector x_F is given in the increasing order of the pair labels[1] of edges in F.

Let $G = (V, E)$ be a subgraph of K_n , that is, $V \subset V_n$ and $E \subset \{(i,j) \in E_n | i,j \in V\}$. Given a graph $G = (V, E)$, for a subset $S \subset V$ we write

$$E(S) = \{ij | ij \in E, i, j \in S\}.$$

Given $u \in R^{p_n}$, $F \subset E_n$,we define,

$$u(F) = \sum_{e \in F} u(e).$$

For any subset S of vertices of V, let $\delta(S)$ denote the set of edges in E with one end in S and the other in $S^c = V \setminus S$.

We say $i \in V$ has *degree* r if the cardinality of $\delta(i) = r$.

A subset H of E is called a *hamiltonian cycle* in G if it is the edge set of a simple cycle in G, of length $|V|$. Or equivalently, $G = (V, H)$ is a subgraph of G which is connected and each vertex has degree 2. We also call such a hamiltonian cycle a $|V|-$*tour* in G. At times we give H by the vector $(1 i_2 \ldots i_{|V|-1} 1)$ where $(i_2 \ldots i_{|V|-1})$ is a permutation of $(2 \ldots |V|)$, corresponding to H. And we call $(1 i_2 \ldots i_{|V|-1})$ a *hamiltonian path* in G.

For details see any standard text on graph theory such as [8].

2.1.1 Forbidden Arcs Transportation (FAT) Problem

A sequence of bipartite graph flow feasibility problems are solved to show the membership in pedigree polytopes, which are defined and studied in section 3.

Let $\mathcal{N} = (\mathcal{V} = O \cup D \ , \mathcal{A})$ be a directed network with node set $\mathcal{V}$, and set of directed arcs $\mathcal{A} = \{(\alpha, \beta) | \alpha \in O \ and \ \beta \in D\}$ with a nonnegative capacity function $c : \mathcal{A} \Rightarrow R_+$. Let O and D be distinct subsets of $\mathcal{V}$; O is the set of sources, D is the set of sinks. For each $\alpha \in O$, we have a nonnegative number a_α, called the supply at α. Similarly, for each $\beta \in D$ we have the demand at β given by a nonnegative number b_β. Let a non negative number, $f_{\alpha\beta}$, called the *flow* along arc $(\alpha, \beta) \in \mathcal{A}$ be defined for each arc. Let $out(\alpha) = \sum_{(\alpha,\beta) \in \mathcal{A}} f_{\alpha,\beta}$ and $in(\alpha) = \sum_{(\beta,\alpha) \in \mathcal{A}} f_{\beta,\alpha}$.

We now consider a special case when for each $\alpha \in O$, we define the set of *forbidden arcs* from α,

$$F(\alpha) \subset \{(\alpha, \beta) \in \mathcal{A} | \text{ for some } \beta \in D\}.$$

Let $F = \cup_{\alpha \in O} F(\alpha)$, denote the set of forbidden arcs. The following problem is called the *forbidden arc transportation* (FAT) problem.

Problem 2.1 *[FAT Problem]*

Find $f_{\alpha\beta} \geq 0 \ \forall \ (\alpha, \beta) \in \mathcal{A} \setminus F$ *such that*

$out(\alpha) = a_\alpha, \quad \alpha \in O$ *and* $in(\beta) = b_\beta, \quad \beta \in D$. *FAT problem is a network feasibility problem, with* $c : \mathcal{A} \to R_+$ *given by*

$$c_{\alpha\beta} = \begin{cases} 0 & if \ (\alpha, \beta) \in F(\alpha) \\ \delta & otherwise \end{cases}$$

where δ *is sufficiently large, say* $\delta = max\{\sum_{\alpha \in O} a_\alpha, \sum_{\beta \in D} b_\beta\}$.

[1] Any pair in E_n is assumed to be in $1-1$ correspondence with its pair label.

This choice of capacity function ensures that there is no flow through arcs belonging to F and there is no real capacity restriction on the flow through other arcs.

The celebrated work by Ford and Fulkerson, *Flows in Networks* [13], is a classic on this subject. For recent developments in bipartite network flow problems see [1].

Next we prove a result, on FAT problems of a special kind, that is crucial for later applications.

Lemma 1 *Given a finite set $\mathcal{D} \neq \emptyset$. Let $g : \mathcal{D} \to Q_+$ be given, with $g(\mathcal{D}) = \sum_{d \in \mathcal{D}} g(d) = 1$. Let $\mathcal{D}^1, \mathcal{D}^2$ be two non empty partitions of $\mathcal{D}$.*

Let $a_\alpha = g(D^1_\alpha), \alpha = 1, \ldots, n_1$ and $b_\beta = g(D^2_\beta), \beta = 1, \ldots, n_2$, where n_k is the cardinality of $\mathcal{D}^k, k = 1, 2$. Let

$$c_{(\alpha,\beta)} = \begin{cases} 1 & \text{if } \mathcal{D}^1_\alpha \cap \mathcal{D}^2_\beta \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\sum a_\alpha = \sum b_\beta = 1 = g(\mathcal{D})$. Then $f_{\alpha\beta} = g(\mathcal{D}^1_\alpha \cap \mathcal{D}^2_\beta) \geq 0$ is a feasible solution for the FAT problem given by (a, b, c).

Proof : $c_{(\alpha,\beta)} \geq f_{\alpha\beta} \geq 0$ can be easily seen. Now,

$$a_\alpha = g(\mathcal{D}^1_\alpha) = \sum_{\beta \ni c_{(\alpha,\beta)}=1} g(\mathcal{D}^1_\alpha \cap \mathcal{D}^2_\beta) = \sum_\beta f_{\alpha\beta}.$$

Similarly, $\sum_\alpha f_{\alpha\beta} = b_\beta$. Hence the feasibility of f. $\square$

3 Pedigree Polytope and MI - Formulation

In this section we present an alternative polyhedral representation of the STSP.

Recall, the cardinality of E_k is given by $p_k = k(k-1)/2$. Let $\tau_n = \sum_{k=4}^{n} p_{k-1}$. Let $\tau_n^{'} = \tau_n - (n-3)$. Let $d_k = p_{k-1} - 1$. So $d_k = p_k - k$.

Let Q_n denote the standard *Symmetric Traveling Salesman Polytope*, $STSP$, given by $Q_n = conv(\mathcal{H}_n)$, where $\mathcal{H}_n$ denotes the set of all *Hamiltonian cycles* (or $n - tours$) in K_n . Given K_n , C:$E_n \to Z_+$, Find $T^* \in \mathcal{H}_n$ such that, $CT^* \leq CT \ \forall \ T \in \mathcal{H}_n$, is called the $STSP$- *optimisation* problem.

Traditionally, in polyhedral combinatorics, Q_n is studied while solving STSP (see [14]). However, we deviate from this and consider an alternative polytope for this purpose. The required notations and concepts follow.

The unique tour, for $n = 3$, having the pairs$\{(1,2),(1,3),(2,3)\}$ is called the $3 - tour$.

Definition 3.1 *Given $T \in \mathcal{H}_{k-1}$, the operation* insertion *is defined as follows: Let $e = (i,j) \in T$. Insert k in e is equivalent to replacing e in T by $\{(i,k),(j,k)\}$ obtaining a $k - tour$. When we denote T as a subset of E_{k-1} then inserting k in T gives us a $T' \in \mathcal{H}_k$ such that,*

$$T' = (T \cup \{(i,k),(j,k)\}) \setminus \{e\}.$$

We write $T \xrightarrow{e,k} T'$.

Given $T \in \mathcal{H}_k$, the operation shrinking *is defined as follows: Let $(\delta(k) \subset T) = \{(i,k),(j,k)\}$. Shrink T is equivalent to replace $\{(i,k),(j,k)\}$ in T by $\{(i,j)\}$ obtaining a $(k-1)-tour$. When we denote T as a subset of E_k then shrink T gives us a $T' \in \mathcal{H}_{k-1}$ such that,*

$$T' = (T \setminus \{(i,k),(j,k)\}) \cup \{(i,j)\}.$$

We write $T \overleftarrow{\ k\ } T'$, and read this as T shrinks to T'.

Notice that shrinking is the inverse operation of insertion. However, by shrinking it is understood that vertex k is chosen for shrinking, but for insertion e needs to be specified.

Definition 3.2 *Let $T \in \mathcal{H}_k$ for $k \in V_n \setminus V_3$. The vector $D(T) = (e_1, \ldots, e_{k-3}) \in E_3 \times \ldots \times E_{k-1}$ is called the* pedigree *of T if and only if $T \in \mathcal{H}_k$ is obtained from the $3-tour$ by the sequence of insertions, viz.,*

$$3-tour \overrightarrow{\ e_1,4\ } T^4 \ \ldots \ T^{k-1} \overrightarrow{\ e_{k-3},k\ } T.$$

The pedigree of T is a compact way of writing T. The pedigree of T can be obtained by shrinking T sequentially to the $3-tour$.

Definition 3.3 *[$X^{(T)}$] Given $T \in \mathcal{H}_n$, let $X^{(T)} = (x_4^{(T)}, \ldots, x_n^{(T)}) \in B^{\tau_n}$ be defined as*

$$\forall k \in V_n \setminus V_3 : x_k^{(T)}(e) = \begin{cases} 1 & \text{if } e = (D(T))_{k-3} \\ 0 & \text{otherwise} \end{cases}$$

$X^{(T)}$ is the characteristic vector of the pedigree of T, where $(D(T))_j = e_j$, the $(j)^{th}$ component of $D(T)$. Let $P_n = \{X^{(T)} \in B^{\tau_n} \mid T \in \mathcal{H}_n\}$.

Consider the convex hull of P_n. We call this the *pedigree polytope*, given by $conv(P_n)$.

Our study is devoted to the discovery of the properties of the pedigree polytope, in order to solve the $STSP$.

3.1 Motivation for MI Formulation

In this section we give briefly the motivation for the MI-formulation of the $STSP$. We can view the $STSP$ as a $(n-3)$ stage decision problem, in which in stage $(k-3), 4 \leq k \leq n$, we have to decide on where to insert k. In the beginning we have the 3-tour. In the first stage, we decide on where to insert 4 among the available pairs $[1,2],[2,3]$, and $[1,3]$. Depending on this decision we have certain available pairs for the second stage insertion.

In general, A_k gives the set of available pairs, $[i,j]$ such that they are adjacent in the $(k-1)$ tour, which results out of the decisions made in the preceding stages. We decide to insert k in an available pair $[i_k, j_k]$. The associated total cost of these decisions made at different stages is

$$\mathcal{C}_{i_4 j_4 4} + \mathcal{C}_{i_5 j_5 5} + \ldots + \mathcal{C}_{i_n j_n n},$$

where $\mathcal{C}_{ijk} = C_{ik} + C_{jk} - C_{ij}$, gives the incremental cost resulting from the insertion of k in (i, j). We are interested in finding optimal $[i_4, j_4], \ldots, [i_n, j_n]$ such that the total cost is minimum. This finally produces an $n-$tour. A $0-1$ programming formulation of the multi stage decision problem described above results in the MI formulation, given in the next section.

3.2 MI Formulation and Tours

Next we state the multistage insertion formulation of the symmetric traveling salesman problem given by Arthanari [2]. Let $u \in R^{p_n}$ be defined for each $e \in E_n$.

Problem 3.4 *[Multistage Insertion]*

$$\begin{aligned}
\textit{minimize} \quad & \mathcal{C}X && (1)\\
\textit{subject to} \quad & && \\
x_k(E_{k-1}) &= 1, k \in V_n \setminus V_3 && (2)\\
\sum_{k=4}^{n} x_k(e) + u(e) &= 1, e \in E_3 && (3)\\
-x_j(\delta(i) \cap E_{j-1}) + \sum_{k=j+1}^{n} x_k(e) + u(e) &= 0, e = (i, j) \in E_{n-1} \setminus E_3 && (4)\\
-\, x_n(\delta(i) \cap E_{n-1}) + u(e) &= 0, e = (i, n) \in \delta(V_{n-1}) && (5)\\
X \in B^{\tau(n)}, \quad & u \in R_+^{p_n} \ . && (6)
\end{aligned}$$

Remark: 1

- *(2) ensures with $0-1$ restriction on X that each k is inserted in exactly one pair.*
- *(3) ensures that each of the pairs $(1, 2), (1, 3)$ and $(2, 3)$ are used for insertion by at most one k.*
- *(4) ensures that each of the other pairs $(1, 4)$ to $(n-2, n-1)$ is used for insertion only when they are available.*
- *(5) defines $u(e)$ for each of the pairs $(1, n)$ to $(n-1, n)$, resulting from all the insertions made in stages 1 through $n-3$.*
- *(5) are redundant, but are included as they define $u(e)$ for those pairs, mentioned above.*

When we allow, X to be real, that is, $X \in R^{\tau_n}$, we refer to the corresponding polytope embedded in $R^{\tau_n + p_n}$ by the name $\mathcal{F}_n$. If $(X, u) \in \mathcal{F}_n$, we say (X, u) is feasible for the $MI - relaxation$. It has been shown that this formulation is equivalent to the subtour elimination formulation, in the sense, that any $u(e), e \in E_n$, feasible for MI -relaxation is feasible for the SEP_n, and given a feasible solution to the SEP_n we have an (X, u) feasible for MI -relaxation (see [5]).

Next lemma establishes the correspondence between $X^{(T)}$ and T through u.

Lemma 2 *Let (X, u) be any integer feasible solution to the multistage insertion problem. Then u is the edge-tour incidence vector of the $n-tour$ T, given by X.*

$$u(e) = \begin{cases} 1 & \textit{if edge } e \textit{ is present in the } n-tour \ \ T \\ 0 & \textit{otherwise} \end{cases}$$

In $STSP$, we are interested in the polytope Q_n whose extreme points are the integer slack variable vectors u corresponding to the integer X in the above formulation. This property of MI formulation, given by lemma 2 also appears in [2] (see [6] for a proof).

In MI -relaxation we have a full description of a polytope that includes P_n. Section 3.1 which deals with the motivation for MI formulation introduced the idea of multistage insertion decisions. Pedigrees correspond to these decisions. As noticed in Remark 1, the constraints (5) are redundant, so dropping them and leaving out the slack variables, we get the polytope designated as $P_{MI}(n)$, given by the equations 2 and the inequalities corresponding to 3 through 4 and the non negativity of X. We state the following theorem without proof (see [4]) which establishes the fact $conv(P_n)$ is the same as the integer hull of the above polytope.

Theorem 3.5

$$P_n \equiv P_{MI}(n) \cap B^{\tau_n}.$$

4 Characterisation Theorems

Our primary interest is to study the membership problem of the pedigree polytope. Alongside we observe the results as applicable to the $STSP$ polytope as well. This is the reason for introducing the MI formulation which bridges the two approaches.

Given $X \in P_{MI}(n)$ and $X/k \in conv(P_k)$, consider $\lambda \in R_+^{|P_k|}$ that can be used as a weight to express X/k as a convex combination of $X^r \in P_k$. Let $I(\lambda)$ denote the index set of positive coordinates of λ. Let $\Lambda_k(X)$ denote the set of all possible *weight vectors*, for a given X and k.

We state the following lemmas without proof.[4].

Lemma 3 *Given $X \in conv(P_n)$, consider $\lambda \in \Lambda_n(X)$. Let T^r be the $n-tour$ corresponding to $X^r, r \in I(\lambda)$. Let $u = \sum_{r \in I(\lambda)} \lambda_r \dot{T}^r$. Then u is the slack variable vector corresponding to X.*

Lemma 4 *Given $u \in Q_n$, then there exists a X such that $(X, u) \in \mathcal{F}_n$, in fact $X \in conv(P_n)$.*

Given a weight vector λ, we define a FAT problem with the following data:

$$\begin{array}{lrcl} O-- & Origins] & : & \alpha, \alpha \in I(\lambda) \\ a-- & Supply] & : & a = \lambda \\ D-- & Sinks] & : & \beta, e_\beta \in E_k, x_{k+1}(e_\beta) > 0 \\ b-- & Demand] & : & x_{k+1} \\ \mathcal{A}-- & Arcs] & : & \{(\alpha, \beta) \in O \times D\} \\ C-- & Capacity] & : & C_{\alpha,\beta} = \begin{cases} \infty & \text{if } \ e_\beta \in T^\alpha \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Recall that we denote the $k-tour$ corresponding to X^α by T^α. We designate this problem as $FAT_k(\lambda)$. Notice that arcs (α, β) not satisfying $e_\beta \in T^\alpha$ are the forbidden arcs, F. We also say FAT_k is feasible if problem $FAT_k(\lambda)$ is feasible for some $\lambda \in \Lambda_k(X)$.

Theorem 4.1 *If $X \in conv(P_n)$ then FAT_k is feasible $\forall\ k \in V_{n-1} \setminus V_3$.*

Proof: Since $X \in conv(P_n)$ we have

$$x_l(e) = \sum_{s \in I(\lambda)} \lambda_s x_l^s(e), e \in E_{l-1}, l \in V_{n-1} \setminus V_3$$

where $I(\lambda)$ is the index set depending on λ for some $\lambda \in \Lambda_k(X)$. To show X is such that FAT_k are all feasible, we shall produce for each $k \in V_{n-1} \setminus V_3$, a λ^k, such that the corresponding $FAT_k(\lambda^k)$ is feasible. We proceed as follows:

First partition $I(\lambda)$ according to the pedigree, $X^\alpha \in P_k$,

$$S_O^\alpha = \{s | s \in I(\lambda) \text{ and } X^s \text{ is a descendant of } \ X^\alpha \in P_k\}. \tag{7}$$

Secondly, partition $I(\lambda)$ according to the edge $e_\beta \in E_k$,

$$S_D^\beta = \{s | s \in I(\lambda) \text{ and } X^s \text{ is such that } \ x_{k+1}^s(e_\beta) = 1\} \tag{8}$$

[O and D in the suffices refer to origins and destinations in the FAT problem.] Let $a_\alpha = \sum_{s \in S_O^\alpha} \lambda_s; b_\beta = \sum_{s \in S_D^\beta} \lambda_s = x_{k+1}(e_\beta)$ and let

$$c_{(\alpha,\beta)} = \begin{cases} \infty & \text{if } S_O^\alpha \cap S_D^\beta \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

In general, let λ^k be defined as $\lambda^k(X^\alpha) = a_\alpha, X^\alpha \in P_k$ and $S_O^\alpha \neq \emptyset$. Then from 7, $\sum a_\alpha = 1$, and $\sum a_\alpha X^\alpha = X/k$. Thus here we have a FAT problem, corresponding to k and λ^k. The feasibility of $FAT_k(\lambda^k)$ for every $k \in V_{n-1} \setminus V_3$ then follows from an application of Lemma 1. Hence the theorem. □

Using Theorem 4.1 and Lemma 4 we have,

Corollary 4.2 *$u \in Q_n \Rightarrow$ there exists a X such that $(X, u) \in \mathcal{F}_n$ and FAT_k is feasible $\forall\ k \in V_{n-1} \setminus V_3$.*

Interestingly, the converse of Theorem 4.1 can be proved [4] and is stated for k below:

Theorem 4.3 *Given $\lambda \in \Lambda_k(X)$, if $FAT_k(\lambda)$ is feasible then $X/k+1 \in conv(P_{k+1})$.*

In Theorem 4.3 we have a procedure to check the membership of $X \in P_{MI}(n)$ in the pedigree polytope, $conv(P_n)$. Since feasibility of a $FAT_k(\lambda)$ problem for a weight vector λ implies the membership of $X/k+1$ in $conv(P_{k+1})$. We can sequentially solve $FAT_k(\lambda^k)$ for each $k = 4, \ldots n-1$ and if $FAT_k(\lambda^k)$ is feasible we set $k = k+1$ and while $k < n$ we repeat; at any stage if the problem is infeasible we stop. So if we have reached $k = n$ we have a proof that $X \in conv(P_n)$.

However if for a $\lambda \in \Lambda_k(X)$ the problem is infeasible we are unable to conclude anything for certain. The feasibility of $FAT_k(\lambda)$ for every $\lambda \in \Lambda_k(X)$ is not required

for $X/k+1$ to be in $conv(P_{k+1})$. An example is given in [4] to illustrate this point. On the other hand we know from Theorem 4.1 that $X/k+1 \in conv(P_{k+1})$ implies that there exists a λ such that $FAT_k(\lambda)$ is feasible.

We can prove, using the recursive structure of the MI -formulation or as corollary to the theorems characterising pedigree polytope, corresponding results for Q_n.

Theorem 4.4 *$u \in Q_n \iff \exists$ a $X \in P_{MI}(n)$ such that (X, u) is feasible for MI- relaxation and FAT_k is feasible $\forall\ k \in V_{n-1} \setminus V_3$.*

5 Conclusions

Pedigree approach for solving $STSP$ is the main theme of this paper. The pedigree polytope is defined and its properties are studied. The motivation for studying this polytope is given. For a $X \in R^{\tau_n}$ to be a member of the pedigree polytope, $conv(P_n)$, it is necessary that X is feasible for the $MI - relaxation$. A partial characterisation of the pedigree polytope, has been achieved in this paper.

The necessity of FAT_k feasibility for all $k \in V_{n-1} \setminus V_3$, for membership in pedigree polytope is proved. For small $n \leq 8$, complete description of $conv(P_n)$ can been achieved, using all possible (full size) FAT_k flow variables and constraints along with $P_{MI}(n)$ description. This approach will not be feasible for larger problems. $FAT_{n-1}(\lambda)$ feasibility for some $\lambda \in \Lambda_{n-1}(X)$, is sufficient for X to be in $conv(P_n)$. Also the recursive nature of the pedigree polytopes, namely, P_n *shrinks* to P_{n-1} can be exploited in checking membership. That is, if $X/k \notin conv(P_k)$ for some $k < n-1$, one need not check any further. However, the complexity of finding a suitable $\lambda \in \Lambda_k(X)$, to give evidence for FAT_k feasibility, has not been established in this paper.

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