A New Positive Dynamic Model Of Capacity Planning With Logical And Binary Set-Up Constraints

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Abstract

One of the most important concepts in production planning is that of the establishment of an overall or aggregate production plan. In this paper we consider this problem for a manufacturing plant. Our aim is to meet the pre-specified demand taking into account decisions concerning when and how many to hire and fire, how much inventory to hold, when to use overtime and undertime, as well as product families and set-up times. We develop a new dynamic model of this issue. The model turns out to be a discrete-time positive system with side linear, and logical and binary set-up constraints. Some interesting new characterizations of production systems important for the objectives of planning and control appear in the model. On the basis of the model we formulate and discuss a discrete-time positive system optimal control problem for capacity planning and provide a number of interesting insights into capacity planning and at the same time address some open problems.

1 Introduction

One of the most important concepts in production planning is that of the establishment of an overall or aggregate production plan. The basic issue is, given a set of production demands stated in some common unit, what levels of resources should be provided in each period? There has been a long history of academic research on aggregate planning, resulting in many (static) mathematical programming models and in a variety of heuristics, see Berry *at al.*[1992]. However, as the firms attempt to implement manufacturing planning and control systems they find serious deficiencies in these models and heuristics. We attempt to overcome some of these drawbacks with a new *dynamic* approach motivated not only by the need for policies but also by the need to incorporate time and open the way for a deductive analysis.

Our aim is to meet a pre-specified demand taking into account decisions concerning product families and the set-up times, when and how many to hire and fire, how much inventory to hold, and when to use overtime and undertime. Some of these characteristics do not appear at all or do not appear together in the models considered in [2, 4]. We develop a *new dynamic model* of the basic issue, which turns out to be a discrete-time positive system with side linear, and logical and binary set-up constraints. Some interesting new characterizations of production systems important for the objectives of planning and control appear in the model. On the basis of this model we formulate and discuss a discrete-time positive system optimal control problem for capacity planning and provide a number of interesting insights into capacity planning and at the same time address some open problems.

2 The Model

We adopt a common unit of production hours. We now introduce the model.

2.1 Dynamics Equations

For $t = 0, 1, 2, \ldots, T-1$, and $i = 1, 2, \ldots, n$,

$$I_{i,t+1} = \beta_{it} I_{it} + \gamma_{it} X_{it} + \delta_{it} O_{it} \tag{1}$$

$$W_{t+1} = \alpha_t W_t + H_t , \tag{2}$$

where

$$0 \le \alpha_t \le 1,\ 0 \le \beta_{it} \le 1,\ 0 \le \gamma_{it} \le 1,\ 0 \le \delta_{it} \le 1 \tag{3}$$

t is the time period (usually a week or a month), n is the number of product families and T is the number of time periods in the horizon of planning, or, simply, the *planning horizon*.

In the difference equations (1)-(2) the *state variables* I_{it} and W_t, the *decision variables* X_{it}, O_{it} and H_{it}, and the *parameters* α_t , β_{it} , γ_{it} and δ_{it} of the production system have the following meaning:

W_t = the number of people employed in month t;
I_{it} = the hours stored in inventory at the end of month t of product family i;
X_{it} = the regular time production hours scheduled in month t for product family i;
O_{it} = the overtime production hours scheduled in month t for product family i;
H_t = the number of employees hired at the end of month t for work in month (t+1);

α_t = the fraction of employees employed in month t that are retained in the month (t+1), the *survival coefficient*;

β_{it} = the fraction of the total of the hours stored in inventory at the end of month t of product family i, which is stored in inventory at the end of month (t+1), the *storage coefficient*;

γ_{it} = the fraction of regular time production hours scheduled in month t which are stored in inventory in month (t+1) of product family i;

δ_{it} = the fraction of overtime production hours scheduled in month t which are stored in inventory in month (t+1) of product family i.

The coefficients α_t (survival), β_{it} (storage), δ_{it} and γ_{it} have an attractive economic interpretation and are quite helpful in the planning process. They are used in the model as exogenous parameters characterizing the production system but their role in the process of decision-making is, clearly, important since they (their values) determine the system evolution. Note also that in (2) $\alpha_t W_t$ is equal to the number of employees employed in month t that are retained in month (t+1), and therefore $(1 - \alpha_t)W_t$ is equal to the number of employees fired in month (t+1). Furthermore, it is not difficult to see from (1) that the hours of production of family i sold in month t is equal to $(1-\beta_{it})I_{it} + (1-\gamma_{it})X_{it} + (1-\delta_{it})O_{it}$.

2.2 Constraints

For t = 0, 1, 2, ..., T -1, and i = 1, 2, …, n,

$$\sum_{i=1}^{n} X_{it} - A_{1t} W_t + U_t + \sum_{i=1}^{n} \varepsilon_i \sigma(X_{it}) = 0 \tag{4}$$

$$\sum_{i=1}^{n} O_{it} - A_{2t} W_t + S_t + \sum_{i=1}^{n} \kappa_i \sigma(O_{it}) = 0 \tag{5}$$

$$I_{it} - B_{it} \geq 0, \tag{6}$$

$$(1 - \beta_{it}) I_{it} + (1 - \gamma_{it}) X_{it} + (1 - \delta_{it}) O_{it} - D_{it} = 0 \tag{7}$$

where

U_t = the number of idle time regular production hours in month t;

S_t = the number of idle time overtime production hours in month t;

B_{it} = the minimum number of hours to be stored in inventory in month t of product family i;

A_{1t} = the maximum number of regular time hours to be worked per employee per month;

D_{it} = the demand for product family i in month t;

A_{2t} = the maximum number of overtime hours to be worked per employee per month;

n = the number of product families;

ε_i = the set-up time for product family i in regular time;

κ_i = the set-up time for product family i in regular time;

$\sigma(X_{it})$ = binary set-up variable for regular time production of product family i in month t;

$\sigma(O_{it})$ = binary set-up variable for overtime production of product family i in month t.

The functions $\sigma(X_{it})$ and $\sigma(O_{it})$ in the constraints (4)-(5) are defined as follows

$$\sigma(z) = \begin{cases} 1, & \text{if} \quad z > 0 \\ 0, & \text{if} \quad z = 0 \end{cases} \tag{8}$$

From their meaning, clearly, all the state and decision variables as well as the parameters introduced above are *non-negative* so that

$$I_{it}, X_{it}, O_{it}, W_t, H_t, U_t, S_t, B_{it}, D_{it}, A_{1t}, A_{2t} \geq 0, \tag{9}$$

The restrictions (4), (5) and (7) on the dynamics of the production system are mixed constraints imposed on the state and decision variables for every time period t. The number U_t of idle time regular production hours in month t , the number of idle time S_t overtime production hours in month t and the minimum number B_t of hours to be stored in inventory in month t are assumed to be exogenous parameters in the model. As a matter of fact the functional constraints (4)-(5) are highly non-linear since they contain the unit step functions (8). Taking into account the set-up times and the product families clearly makes the model much more complicated.

2.3 Boundary Conditions

$$W_o = A_3 \geq 0 \tag{10}$$
$$I_{io} = A_{i4} \geq 0 \tag{11}$$
$$W_T = A_5 \geq 0 \tag{12}$$
$$I_{iT} = A_{i6} \geq 0 \quad , i = 1, 2, \ldots , n, \tag{13}$$

where

A_3 = the initial employment level;

A_{i4} = the initial inventory levels of product family i;

A_5 = the desired number of employees in month T (the last month of the planning horizon);

A_{i6} = the desired inventory level of product family i at the end of month T.

The states W_o and I_o are called initial states, and the states W_T and I_{iT} are final (terminal) states.

2.4 Assumptions

The dynamic model (1)-(13) described above is introduced under the following assumptions. In any month t:

- All regular time employees work overtime.

- Only existing regular time employees work overtime.
- All employees work the same number of regular time hours, up to the limit A_{1t}.
- All employees work the same number of overtime hours, up to the limit A_{2t}.

The dynamic model for capacity planning (1)-(13) can be built-in in a decision support system. It is somewhat easier for simulation and decision-making than the static models considered in {2, 7]. On the other hand, introducing an objective (cost) function we can consider the related optimal control problem and determine the optimal decision sequences and the corresponding optimal state trajectory over the horizon of planning T. Such an optimal control problem is formulated and discussed in the section 4.

3 Positive Linear System Dynamics

The dynamic equations (1)-(2) can be rewritten in the matrix form

$$\begin{bmatrix} W_{t+1} \\ I_{1,t+1} \\ \vdots \\ I_{n,t+1} \end{bmatrix} = \begin{bmatrix} \alpha_t & 0 & \cdots & 0 \\ 0 & \beta_{1t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{nt} \end{bmatrix} \begin{bmatrix} W_t \\ I_{1t} \\ \vdots \\ I_{nt} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_{1t} & 0 & \cdots & 0 & \delta_{1t} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{nt} & 0 & 0 & \cdots & \delta_{nt} \end{bmatrix} \begin{bmatrix} H_t \\ X_{1t} \\ \vdots \\ X_{nt} \\ O_{1t} \\ \vdots \\ O_{nt} \end{bmatrix}, \tag{14}$$

$$t = 0, 1, 2,..., T\text{-}1,$$

or, respectively,

$$\boldsymbol{x}(t+1) = \boldsymbol{A}(t)\, \boldsymbol{x}(t) + \boldsymbol{B}(t)\, \boldsymbol{u}(t), \qquad t = 0, 1, 2,..., T\text{-}1, \tag{15}$$

where the vector of state variables $\boldsymbol{x}(t)$, the decision (control) vector $\boldsymbol{u}(t)$, the system matrix $\boldsymbol{A}(t)$ and the control matrix $\boldsymbol{B}(t)$ are given by the corresponding vectors and matrices in (14). Note that all of the entries of $\boldsymbol{u}(t)$, $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$ are greater than or equal to zero for any time period t. Vectors and matrices with nonnegative entries are called nonnegative vectors and matrices, see Berman and Plemmons [1994]. They are denoted as $\boldsymbol{u}(t) \geq 0$ and $\boldsymbol{A}(t) \geq 0$, respectively. Since the system matrix $\boldsymbol{A}(t) \geq 0$, the control matrix $\boldsymbol{B}(t) \geq 0$ and the decision vector $\boldsymbol{u}(t) \geq 0$ are nonnegative for any t it can be seen from (14) (or (15)) that the state vector $\boldsymbol{x}(t)$ is a nonnegative vector whenever the initial state

$$\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_{n+1}(0) \end{bmatrix} = \begin{bmatrix} W_o \\ I_{1o} \\ \vdots \\ I_{no} \end{bmatrix} = \begin{bmatrix} A_3 \\ A_{1,4} \\ \vdots \\ A_{n,4} \end{bmatrix} \tag{16}$$

is nonnegative. Thus, nonnegativity (positivity) is an intrinsic property of the system (14), that is (15). Such systems are called *positive linear systems*, see, for example Luenberger [1979]. It can be proved that the conditions $\boldsymbol{u}(t) \geq 0$, $\boldsymbol{A}(t) \geq 0$ and $\boldsymbol{B}(t) \geq 0$ are necessary and sufficient for the state trajectory $\{\boldsymbol{x}(t)\}$ to be nonnegative for any t.

Note also that the nonnegativity of the decision variables $u_1(t) = H_t$, $u_{2i}(t) = X_{it}$ and $u_{3i}(t) = O_{it}$ guarantees the nonnegativity of the state variables in the functional constraints (4), (5) and (7). Note also that the final (terminal) state is

$$\mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_{n+1}(T) \end{bmatrix} = \begin{bmatrix} W_T \\ I_{1T} \\ \vdots \\ I_{nT} \end{bmatrix} = \begin{bmatrix} A_5 \\ A_{1,6} \\ \vdots \\ A_{n,6} \end{bmatrix}$$

is non-negative too.

The (dynamic) system theory for positive linear systems has been rapidly developing during the last decade although one of the cornerstones of this theory is the famous Frobenius-Perron theorem for nonnegative matrices known for over 80 years, see Berman and Plemmons [1994] or Luenberger [1979]. The Frobenius-Perron theorem plays a fundamental role in mathematical economics, input-output analysis, economic dynamics, probability theory and mathematical statistics, and any linear theory involving positivity.

4 The Optimal Control Problem

A relevant functional (objective function, criterion, cost function, performance index) can be the cost of our decisions about the regular time production hours of product family *i* scheduled in month *t*, the overtime production hours of product family *i* scheduled in month *t*, the number of employees hired at the end of month *t* for work in month (*t*+1), the number of employees fired in month *t* and the related work force as well as the inventory expenses and the set-up costs of product family *i* in regular time and, respectively in overtime. Thus

$$z = \sum_{t=1}^{m}[(c_H H_t + c_F(1-\alpha_t)W_t + A_{1t}c_R W_t + A_{2t}c_O W_t + c_U U_t + c_S S_t) +$$

$$+ \sum_{i=1}^{n}(c_{iR}X_{it} + c_{iO}O_{it} + c_{iI}I_{it} + c_{iRS}\sigma(X_{it}) + c_{iOS}\sigma(O_{it}))] \qquad (16)$$

where

c_H = the cost of hiring an employee;
c_F = the cost of firing an employee;
c_R = the regular time work force cost per employee hour;
c_O = the overtime work force cost per employee hour;
c_U = the cost per labour hour of idle regular time production;
c_S = the cost per labour hour of idle overtime production;
c_{iR} = the cost per labour hour of regular time production of product family *i*;
c_{iO} = the cost per labour hour of overtime production of product family *i*;
c_{iI} = the cost per month of carrying on labour hour of work of product family *i*;
c_{iRS} = the set-up cost of product family *i* in regular time;
c_{iOS} = the set-up cost of product family *i* in overtime.

Note that the unit costs might depend on t.

We formulate now the discrete-time optimal control problem.

Problem OCP

minimize z

s.t.

difference equations (1)-(2)

constraints (4)-(9)

boundary conditions (10)-(13).

The dynamics equation (1)-(2), the objective function (16) and the constraints (6)-(7) are linear, but the functional constraints (4)-(5) are highly non-linear since they contain the unit step functions (8). Moreover, (4)-(5) and (7) are constraints imposed on the decision and state variables, that is mixed state-space and control constraints. On the other hand the positivity property of the system (14) implies that the nonnegativity constraints (9) on the states $x_1(t) = W_t$ and $x_{i+1}(t) = I_{it}$, $i = 1,..., n$, are automatically satisfied for any non-negative decision sequences $\{H_t\}$,$\{X_{it}\}$and$\{O_{it}\}$. The initial and terminal (final) points(10)-(13) of the state trajectory are fixed so the optimal control problem (Problem OCP) formulated above is a *two-point boundary-value nonlinear dynamic optimization problem* with mixed state-space and control constraints. The solution to this problem is the optimal decision sequences of hiring $\{u_1(t)\}=\{H_t\}$, regular time production hours of product family i $\{u_{2i}(t)\}=\{X_{it}\}$and overtime production hours of product family i $\{u_{3i}(t)\}=\{O_{it}\}$, and the corresponding optimal trajectories of the number of people $\{x_1(t)\} = \{W_t\}$ and the hours stored in inventory$\{x_{i+1}(t)\} = \{I_{it}\}$, $i = 1,..., n$, which minimize the cost functional (16).

The optimal control approach to the theory of the firm is motivated by three issues: (i) the need for policies, (ii) the contribution of deductive analysis, and (iii) the need to incorporate time. Van Hilton *at al* [1993] have well exposed the state-of-the art of this area but they discuss only continuous-time systems and exploit Pontryagin Maximum Principle developed for such systems. They do not consider positive systems as well as discrete-time models. Discrete-time models are somewhat more suitable to describe the firm's dynamics. Moreover, the model (1)-(13) not only represents a discrete-time positive system but it contains a number of important parameters not included in the dynamic models described in the literature. At the same time, the first question that arises when solving any two-point boundary-value optimal control problem is whether there exists an admissible decision sequence $\{u(t)\}$that can carry out the production system from the given initial state $\boldsymbol{x}(0)$ into the specified final state (goal) $\boldsymbol{x}(T)$. This question is closely related to the (positive) controllability properties of the system. Unfortunately not much attention to date is paid to controllability of the dynamic models of the firm as it is evident from Luenberger [1979} and van Hilton *at al* [1993]. We have studied the (positive) controllability properties of the production system (1)-(3) in a related paper [3].

In the next section we give some computational considerations for solving the optimal control problem OCP.

5 Some Computational Considerations

The unit step functions $\sigma(X_{it})$ and $\sigma(O_{it})$ in the constraints (4)-(5) are the main obstacle in solving Problem OCP by using a general optimal control solver. To overcome the problem creating by the discontinuity of the unit step functions given by (8) in the constraints (4)-(5) we approximate them as follows

$$\sigma(z) \approx 1 - e^{-Mz}, \quad 0 \leq z < \infty, \tag{17}$$

where $M > 0$ is a sufficiently large experimentally determined constant. Note that the function $\sigma(z) \to 1$ as $z \to \infty$, and $\sigma(z) \to 0$ as $z \to 0$. After the substitution of (17) in the constraints (4)-(5) and the objective function (16) they become continuous functions in X_{it} and O_{it} so that this makes it possible to use a general optimal control solver for solving the discrete-time optimal control problem (OCP).

We have used DMISER 3 [5] to solve several instances of Problem (OSP) and obtain a solution to the approximated problem. DMISER 3 is a software package for solution of combined optimal control and optimal parameter selection problems in which the system dynamics is described by difference equations. It is modeled on MISER 3 [5]. Even though the controls in discrete-time problems are naturally parametrized, DMISER 3 exploits the idea of control parametrization. According to the authors of the package the representation of individual controls (decisions) over a number of time steps by a single parameter greatly reduces the number of decision variables in problems with a large number of time steps as well as improving the likelihood of convergence to an optimal solution. The procedure of installing and executing the DMISER 3 as well as the user subroutines and output files can be found on *http://cado.maths.uwa.edu.au/miser/*.

DMISER 3 is a general purpose software package. The structure of the discrete-time optimal control problem (OCP) formulated above allows for the development of more efficient specialized exact algorithms. We are now in the process of developing such an algorithm.

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