Paradox of Multiple Election – the Probabilistic Approach

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Abstract.
The probability of aggregation paradox for aggregation by alternatives and aggregation by combinations is evaluated as well as calculation estimation is done. General hypothesis about tendency to paradoxical results is formulated.
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1 Introduction

The literature of statistics and social choice contains descriptions of many types of aggregation paradoxes. For example: Anscombe's paradox (Wagner 1984) or Ostrogorski's paradox (Kelly 1989). The paradox analysed here is an aggregation paradox, which occurs in a specific case: multiple elections in which voters may not know results of one election before they vote another (Brams 1993). In a statistical sense results of one vote are independent on results of another vote. The analysis of the paradox may be applicable in referendum containing several alternatives to vote for or against, or in sequential votes in legislature.

The set of winner's stances in each of the individual elections is called the winning combination. In some cases it is possible that nobody vote for winning combination, but the combination is still winning one. This phenomenon is called the paradox of multiple elections (Brams 1993). In this paper the chance for occurrence of the paradox of multiple election is evaluated and computed.

2 Example of the paradox

Let R be referendum containing 3 alternatives. Voters can vote on each alternative for (=1) or against (=0). In this case we have $2^3 = 8$ combinations of individual outcomes. The paradox occurs when the winning combination receives the fewest votes.

Assume that the winning combination is 111. Let n=10 denote number of voters. Suppose that 10 voters give the following outcomes:

Combination	Number of votes
111	0
110	2
101	2
011	4
100	2
010	0
001	0
000	0

Each alternative receives 6 votes for, 4 votes against. Winning combination is 111 but no voter prefers to vote 111. In aggregation by alternatives (Brams 1993, Holubiec, Mercik 1994) the winning combination is 111 however, in aggregation by combination this combination receives 0 votes.

3 Probabilistic model of individual voting preferences

Let *n* be the number of voters. Denote $X_1, X_2, ..., X_n$ as individual results of vote. Let *m* denote the number of alternatives to vote for or against in one vote. We assume that results of voting are independent on voters and on alternatives. For example, result denoted by 1011 when *m*=4, corresponds to the following situation: alternatives 1, 3 and 4 “for” and alternative 2 “against”.

The result of individual vote may be analysed as random variable. Such variable has binomial distribution. Hence, $X_1, X_2, ..., X_n$ denote set of m-dimensional random variables. Let

$$
\begin{aligned}
&X_1 = (X_{11}, X_{12}, X_{13}, ..., X_{1m}),\\
&X_2 = (X_{21}, X_{22}, X_{23}, ..., X_{2m}),\\
&X_3 = (X_{31}, X_{32}, X_{33}, ..., X_{3m}),\\
&....\\
&X_n = (X_{n1}, X_{n2}, X_{n3}, ..., X_{nm}),
\end{aligned}
$$

where random variable X_{ij} has distribution given by following probability function: $P(X_{ij} = 1) = p_i$, $P(X_{ij} = 0) = 1 - p_i$ for $i = 1,2,...,n$, $j = 1,2,...,m$.

In that case p_i denote probability of voting for (stance 1). We interpret this as follows: when p_i equals to 1 or 0 the given voter is fully determined and he or she votes for stance 1 or 0 respectively. If all voters are determined in the above sense probability (actually conditional probability) of occurrence of paradox is equal to 1 (paradox does appear) or 0 (paradox does not appear). In all other situations the paradox has probability in the middle of (0,1) interval.

The probability p_i may be different for each voter. Probability p_i corresponds to individual preferences of voters. *N*-dimensional vector $p = (p_1, p_2, p_3, ..., p_n)$ describe general electoral preferences. The distribution of p_i can be introduced as a measure of nondetermination of the electorate. In some sense, the distribution of p_i along 0 – 1 dimension can be viewed as electorate density along the same normalised dimension. Every determined voter gives the non-paradoxical results – he or she never gives her or his vote for one stance for given alternative and, at the same moment, he or she gives vote for concurrent stance for another alternative. Only non-deterministic voters produce paradoxes – their position along 0 – 1 dimension is somewhere in the inside of the interval (0,1).

Let us assume that vector p is *n*-dimensional random variable, where each p_i has Beta distribution given by density

$$
f(p_i) = \begin{cases} \dfrac{1}{B(w,s)} p_i^{w-1}(1-p_i)^{s-1} & for \quad p_i \in (0,1) \\ 0 & for \quad p_i \notin (0,1) \end{cases},
$$

where $B(w,s) = \int_0^1 x^{w-1}(1-x)^{s-1}dx \quad for \quad w,s > 0$.

Parameters w, s determine shape of curve of density along 0 – 1 dimension, these parameters determine balance of voters along 0 – 1 dimension. For example if $w, s = 1$ each p_i has unimodal distribution; every result of vote has identical probability

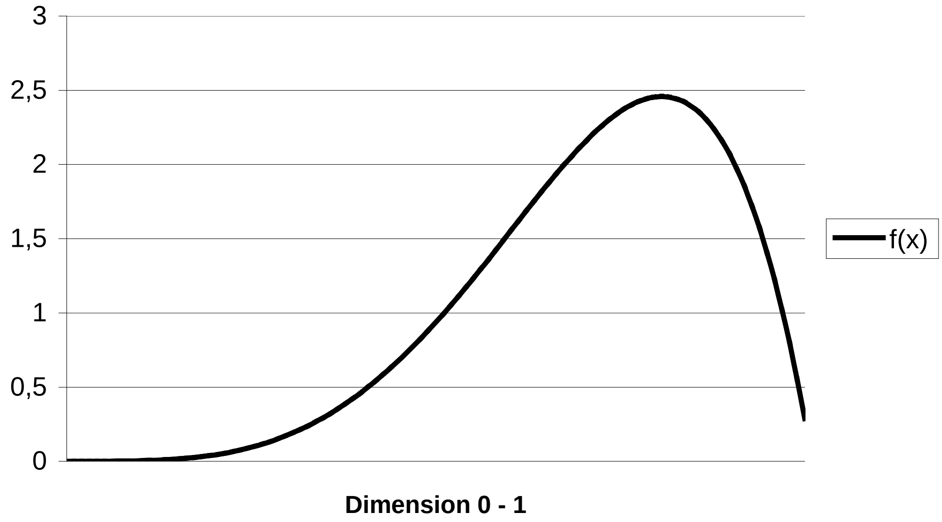

Figure 1. B(5,2) probability density function.

equals to ½. If $w = 5$ and $s = 2$ electorate preferences are given by probability density shown on figure 1.

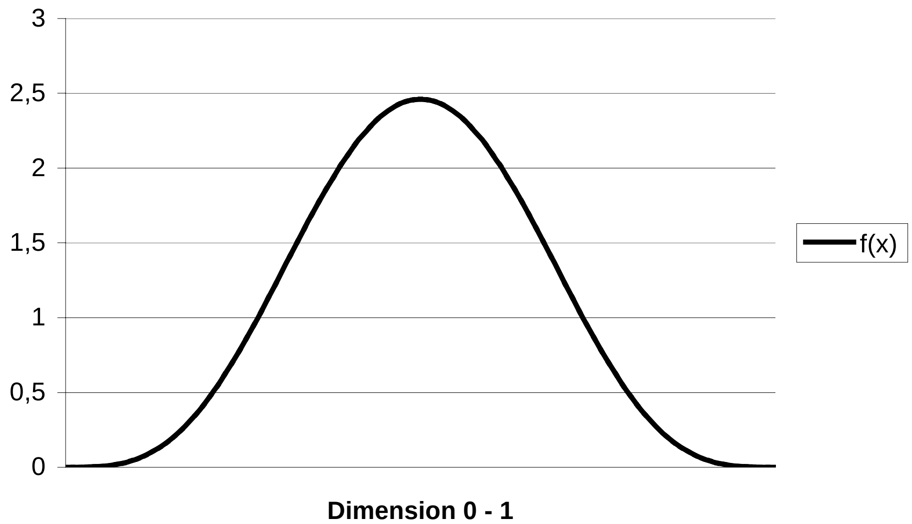

Figure 2. B(5,5) probability density function.

For any given i, j (i - number of voters, j - number of alternatives) we have:

$$P(X_{ij} = 1) = \int_0^1 p_i \frac{1}{B(w,s)} p_i^{w-1}(1-p_i)^{s-1} dp_i = \frac{B(w+1,s)}{B(w,s)} = p \qquad (*)$$

$$P(X_{ij} = 0) = 1 - \frac{B(w+1,s)}{B(w,s)} = 1 - p\ .$$

4 Chances of the wining combination

Let combination 111..1 is the winning combination in aggregation by alternatives. Denote *n* – number of voters, *m* – number of alternatives. Let decision rule is simple majority. Chances for the combination 111...1 to became the winning combination are given by following probability:

$$P(111...11\, winner) = P(\sum_{i=1}^{n} X_{i1} > \frac{n}{2}, \sum_{i=1}^{n} X_{i2} > \frac{n}{2}, \sum_{i=1}^{n} X_{i3} > \frac{n}{2}, ..., \sum_{i=1}^{n} X_{im} > \frac{n}{2}).$$

Each random variable $\sum_{i=1}^{n} X_{ik}$ for any $k = 1,2,...,m$ has Bernoulli distribution with parameters *n* and p, where p denotes probability given by equation (*). From central limit theorem one can find approximation:

$$P(\sum_{i=1}^{n} X_{i1} > \frac{n}{2}, \sum_{i=1}^{n} X_{i2} > \frac{n}{2}, \sum_{i=1}^{n} X_{i3} > \frac{n}{2}, ..., \sum_{i=1}^{n} X_{im} > \frac{n}{2}) \approx \left(1 - \Phi\left(\frac{\frac{n}{2} - np}{\sqrt{np(1-p)}}\right)\right)^{m},$$

where $\Phi(.)$ denotes distribution function of standardised normal distribution.

In the same way one can find probability that combination 010...1 is the winning combination in the aggregation by alternatives. For example if *m*=7 probability for combination 1010110 to be a winner can be estimated by

$$P(1010110\, winner) =$$

$$= P(\sum_{i=1}^{n} X_{i1} > \frac{n}{2}, \sum_{i=1}^{n} X_{i2} < \frac{n}{2}, \sum_{i=1}^{n} X_{i3} > \frac{n}{2}, \sum_{i=1}^{n} X_{i4} < \frac{n}{2}, \sum_{i=1}^{n} X_{i5} > \frac{n}{2}, \sum_{i=1}^{n} X_{i6} > \frac{n}{2}, \sum_{i=1}^{n} X_{i7} < \frac{n}{2})$$

$$\approx \left(1 - \Phi\left(\frac{\frac{n}{2} - np}{\sqrt{np(1-p)}}\right)\right)^{4} \cdot \left(\Phi\left(\frac{\frac{n}{2} - np}{\sqrt{np(1-p)}}\right)\right)^{3}$$

This approach enables to compute probability for any given combination to be winning combination in aggregation by alternatives.

5 Chances of appearance of paradox

Paradox occurs when given combination is the winner in aggregation by alternatives, but receives the fewest number of votes (or not the highest number of votes) in aggregation by combination. This is the reason for modification of the approach to computation of probability of paradox.

Let *C* denote a given combination. Now we construct necessary and sufficient condition for *C* to be combination, which wins in aggregation by alternatives, and loses in combination aggregation. Let $n(C)$ denotes number of votes given for combination *C* in combination aggregation. The number $n(C)$ actually describes number of voters vote according to stance given by combination *C*. In general approach, in one multiple election there are possible 2^m combinations. Let $C_1, C_2, C_3, ..., C_{2^m}$ denote all possible combinations in given multiple election. Then $C_l = \{(c_{l1}, c_{l2}, ..., c_{lm})\}$ for $l = 1,2,...,2^m$, where $c_{li} = 0,1$ for *i=1,2,...,m*. For example if *m*=3 we have $C_1 = \{(1,1,1)\}$,

$C_2 = \{(1,1,0)\}$, $C_3 = \{(1,0,1)\}$, $C_4 = \{(0,1,1)\}$, $C_5 = \{(1,0,0)\}$, $C_6 = \{(0,1,0)\}$, $C_7 = \{(0,0,1)\}$, $C_8 = \{(0,0,0)\}$.

Obviously total number of votes is distributed over 2^m combinations

$$\sum_{k=1}^{2^m} n(C_k) = n .$$

Paradox could appear if

$$\forall k = 1,2,...,2^m \quad n(C) \le \min\{n(C_1), n(C_2), n(C_3),...., n(C_{2^m})\},$$

where C denotes winning combination in aggregation by alternatives.

The above condition may be modified to the following one

$$n(C) = 0 ,$$

where combination C receives 0 votes or

$$n(C) < \max\{n(C_1), n(C_2), n(C_3),...., n(C_{2^m})\}$$

when combination C is not winning combination in combination aggregation.

Each voter (total number of voter equals to n) prefers one combination from the set of possible combinations. Results of voting are independent.

Let $1(C)$ counts how many alternatives' codes equal 1 are in combination C. Probability, that given voter votes for combination C may be computed as follows

$$P_C = P((X_{i1}, X_{i2}, X_{i3},..., X_{im}) = C) = p^{1(C)}(1-p)^{m-1(C)},$$

where p denotes probability given by equation (*).

In combination aggregation the result of multiple election may be analysed as 2^m - dimensional random variable (random vector). Under assumption, that results of votes are independent, the random vector has polynomial distribution with parameters: n - number of voters, $p_{C_1}, p_{C_2}, p_{C_3},..., p_{C_{2^m}}$ - probabilities of voting for combinations $C_1, C_2, C_3,..., C_{2^m}$.

Let $N(C)$ be random variable, which counts how many votes are given for combination C. In this situation random vector $(N(C_1), N(C_2), N(C_3),...., N(C_{2^m}))$ describes the results of multiple election. Then

$$P(N(C_1) = n(C_1), N(C_2) = n(C_2),...., N(C_{2^m}) = n(C_{2^m})) = \frac{n!}{n(C_1)!n(C_2)!...n(C_{2^m})!} p_{C_1}^{n(C_1)} p_{C_2}^{n(C_2)} ... p_{C_{2^m}}^{n(C_{2^m})}$$

where $n(C_l)$ is value of random variable $N(C_l)$ for $l = 1,2,...,2^m$; $\sum_{i=1}^{2^m} n(C_i) = n$ and $\sum_{i=1}^{2^m} p_{C_i} = 1$.

For any given combination $C = (c_1, c_2,..., c_m)$ probability of paradox occurrence connected to this combination equals to

$$P(paradox\, C) = \sum_{n(C) \le \min\{C_i\} \wedge \forall i=1,2,...,m \sum_{k=1}^{2^m} n(C_k) \cdot c_{ki}^* > \frac{n}{2}} P(N(C_1) = n(C_1), N(C_2) = n(C_2),...., N(C_{2^m}) = n(C_{2^m})),$$

where $c_{ki}^* = \begin{cases} 1 & dla \quad c_{ki} = c_i \\ 0 & dla \quad c_{ki} \ne c_i \end{cases}$ for $k = 1,2,3,...,2^m$, $i = 1,2,3,..., m$.

Occurrence of the paradox connected to one combination is independent on paradox connected to another combination. Probability of occurrence paradox (no matter which combination is winning one) is given by equation

$$P(paradox) = \sum_{l=1}^{2^m} P(paradox\, C_l)\,.$$

6 Empirical Examples.

6.1 Small set of voters and symmetrical preference's distribution.

Let us consider multiple election containing 3 alternatives. Let us assume that number of voters equals 20. Now *n*=20, *m*=3. Let *w*=1 *s*=1 describe electoral preferences. In this case probability density function of each parameter p_i is unimodal. It means that voters do not prefer any special stances in any alternative; probability to vote "for" for any alternative equals exactly to ½.

Combination	Probability of the paradox	Probability of being winner in aggregation by alternatives
111	0,00035	0,125
110	0,00035	0,125
101	0,00035	0,125
011	0,00035	0,125
100	0,00035	0,125
010	0,00035	0,125
001	0,00035	0,125
000	0,00035	0,125

Table 1. Results of computation for symmetrical preferences of voters. Rounded values (five decimal places).

In this case probability of the paradox: $P(paradox) = 0{,}00278$.

6.2 Small set of voters and asymmetrical preference's distribution.

One of the asymmetric distributions of preferences presents B(5,20) probability density function.

Let *n*=20, *m*=3, *p*=0,2. In this case probability of paradox: $P(paradox) = 0{,}000000384602667$.

Combination	Probability of paradox (p=0,2, B(5,20))	Prob. of winner in aggregation by alternatives (p=0,2 B(5,20))	Probability of paradox (p=0,8, B(20,5))
111	0,000000000004128	0,000000000063126	0,000000353781230
110	0,000000000241461	0,000000158476115	0,000000010030975
101	0,000000000241461	0,000000158476115	0,000000010030975
011	0,000000000241461	0,000000158476115	0,000000010030975
100	0,000000010030975	0,000397852848408	0,000000000241461
010	0,000000010030975	0,000397852848408	0,000000000241461
001	0,000000010030975	0,000397852848408	0,000000000241461
000	0,000000353781230	0,998805965963306	0,000000000004128

Tab. 2. Results of computation for asymmetrical preferences of voters. Rounded value (fifteen decimal places).

We can easy obtain results of computation in situation described by set of parameters n=20, m=3 and p=0,8, because probabilities of paradoxes connected to combinations are symmetrically distributed according to results in Tab. 2 and probability of paradox is exactly the same.

6.3 Medium set of voters.

Let n=50 and m=3.

Combination	Probability of paradox (p=0,5, B(1,1))	Probability of winner in aggregation by alternatives (p=0,5, B(1,1))	Probability of paradox (p=0,05, B(2,40))	Probability of winner in aggregation by alternatives (p=0,05, B(2,40))
111	0,000409	0,125	2,44E-64	0
110	0,000409	0,125	2,67E-53	0
101	0,000409	0,125	2,67E-53	0
011	0,000409	0,125	2,67E-53	0
100	0,000409	0,125	9,40E-48	0
010	0,000409	0,125	9,40E-48	0
001	0,000409	0,125	9,40E-48	0
000	0,000409	0,125	1,36E-43	1
Total	0,003272	1	1,36E-43	1

Table 3. Results for 50 voters.

7 Conclusions

The probability of the paradox is the highest one for all situations with symmetrical distribution of an electorate. Then probability of being the winner for given combination under the alternatives aggregation is equally distributed between every combinations. This applies also to probabilities of paradoxes connected with given combinations. Higher asymmetry of voters' preference determines decreasing of probability of the paradox and step by step concentration of probability of being a winner in aggregation by alternatives around eccentric combinations (0,0,0) or (1,1,1). Also in this case probabilities of the paradox for one of those combinations dominate over other probabilities. Hence, we have deepening asymmetry of the probability distribution.

Results obtained in computational experiments let us to formulate the ***general hypothesis:***

> ***The probability of multiple election paradox for two stances alternative increases with increasing of balance of distribution of voter's preferences if we define a distribution being the most balanced when median value for this distribution is equal 1/2.***

Because in real voting situation (when number of voters is significantly great) the paradox of multiple election is observe relatively often we suppose that electorate distribution of tendency toward one from two hypothetical stances 0 or 1 is symmetrical. In other words, the society is symmetrically distributed (in the sense of symmetrical preferences) along two-dimensional political dimension created in a sense of given alternative. Any disturbance in balance of the society effects by disappearing of the multiple election paradox.

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